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Delta-Rho, A Perturbation Method for General and Astrodynamical Trajectories

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The Delta-Rho perturbation technique is developed in a form applicable to both general dynamical and astrodynamical trajectories. It is subsequently verified for a specific astrodynamic example. It can be described as an extension of Encke's perturbation method. Like Encke it uses analytically determined trajectories, but unlike Encke it requires the definition and storage of a reference trajectory. If the reference trajectory includes all of the important perturbing forces to be considered, the integration of the state equations for the dispersed trajectories can yield satisfactory results when implemented using a highly simplified perturbation-force model. The Delta-Rho method is particularly useful when considering trajectories in the neighborhood of a reference ("nominal") trajectory and has potential application to Monte Carlo studies of spacecraft missions, development of real-time guidance laws, and real-time generation of satellite ephemerides in small, vehicle-borne computers, to name a few.

I. Introduction

DELTA-Rho is a new method for propagating perturbed trajectories of nonlinear systems.[1,2] In astrodynamics, it belongs to the class of special perturbations, a class which includes Cowell's and Encke's methods. Whereas traditionally Cowell's and Encke's methods are applied to problems involving the three-degree-of-freedom translational motion of a point mass in a nonideal central force field, here we shall present a generalized treatment of these methods, followed by the derivation of Delta-Rho in the same pattern, to illustrate their relationship to one another as a family of methods, as well as to suggest their use for problems outside of astrodynamics. We shall find that all three methods apply to general trajectories, where the latter are considered to be the solution of the differential equations of a dynamical system in state-vector form. Then we shall illustrate the effectiveness of the Delta-Rho method in examples chosen from astrodynamics.

II. Formulations Applicable to General Dynamical Problems

A. Cowell's and Encke's Methods

Let $X(t)$ denote the n-vector state of a continuous deterministic system and assume that $X(t)$ is defined in state-variable form. Then

$$\dot{X}(t) = f[X(t)] \tag{1}$$

Suppose that $f[\cdot]$ is a vector function which is comprised of two vector functions $g[\cdot]$ and $\gamma[\cdot]$ such that

$$\dot{X}(t) = f[X(t)] = g[X(t)] + \gamma[X(t)] \tag{2}$$

$$\|\gamma[X(t)]\| \ll \|g[X(t)]\| \tag{3}$$

Then $g[\cdot]$ is called the principal part of Eq. (2) while $\gamma[\cdot]$ is the perturbation. Suppose further that there exists an analytically determined n-vector state $x(t)$ which is a known solution to the principal part, that is,

$$\dot{x}(t) = g[x(t)] \tag{4}$$

with initial conditions

$$x(t_i) \equiv X(t_i) \tag{5}$$

for a reference time t_i usually referred to as the rectification time. In this case the trajectory $X(t)$ is considered a perturbation of the trajectory $x(t)$, the latter being called the "osculating" trajectory (Fig. 1).

It is convenient to define an n-vector $\rho(t)$ as the difference

$$\rho(t) \equiv X(t) - x(t) \tag{6}$$

Differentiation yields

$$\dot{\rho}(t) = \dot{X}(t) - \dot{x}(t) \tag{7}$$

Substitution of Eqs. (2) and (4) into Eq. (7) yields

$$\dot{\rho}(t) = g[X(t)] - g[x(t)] + \gamma[X(t)] \tag{8}$$

Cowell's method consists of the forward numerical integration of Eq. (1) or (2) from the initial condition [Eq. (5)]. Encke's method consists of the numerical integration of the state equation (8) given the initial condition $X(t_i)$ n-vector for some reference time t_i, and the algebraic addition of the resulting $\rho(t)$ to the analytically determined osculating trajectory $x(t)$ according to the trajectory combination equation

$$X(t) = x(t) + \rho(t) \tag{9}$$

As the $\rho(t)$ and $X(t)$ trajectories are propagated, if $\|\rho(t)\|$ becomes greater than some error criterion ϵ, the process can be rectified by reinitializing the osculating trajectory $x(t)$ to match $X(t)$, that is, set

$$x(t_{i+1}) \equiv X(t_{i+1})$$

for some time $t_{i+1} > t_i$.

B. Derivation of the Delta-Rho Equations; The Delta-Rho Method

Unlike Cowell's and Encke's methods, the Delta-Rho method assumes that an "exact" reference trajectory $[X_0(t)]$ is known and is available to be used in the required computations. This reference trajectory can be a nominal trajectory about which several dispersed trajectories are to be

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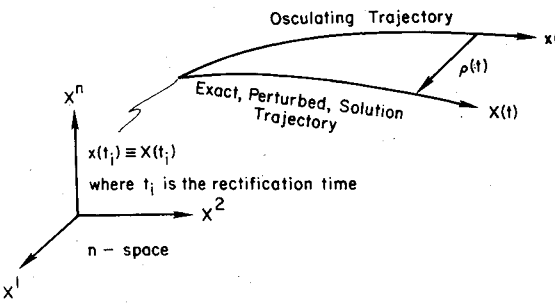

Fig. 1 Generalized Encke's method geometry.

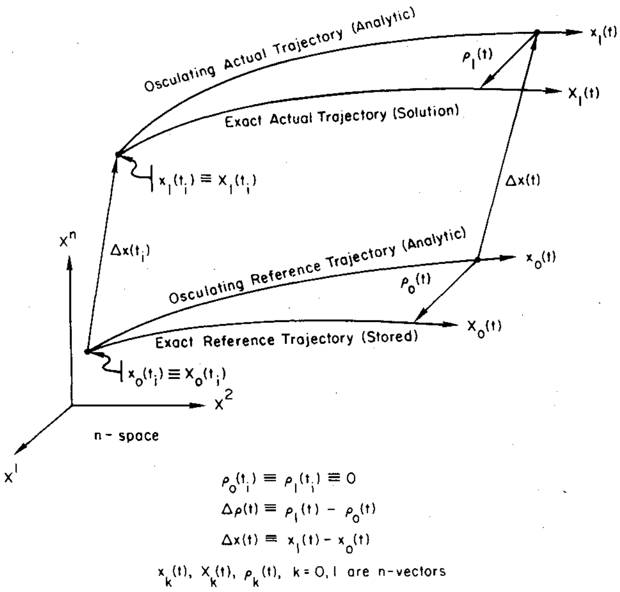

Fig. 2 Delta-Rho geometry.

determined, for example, as in a Monte Carlo study. It is further assumed that the analytically derived trajectories do not differ greatly from the "exact" trajectories, that is, that the "exact" trajectories can be considered as a perturbation from the analytic trajectories. The analytic trajectories are obtained as a function of an initial state (i.e., the state at the current rectification time t_i). With reference to Fig. 2, let

$X_0(t)$ be the exact reference trajectory which is known and stored for use in the computations, with initial state $X_0(t_i)$

$x_0(t)$ be the analytically derived trajectory with the same initial state, that is $x_0(t_i) \equiv X_0(t_i)$

$X_1(t)$ be the solution (exact) trajectory with initial state $X_1(t_i)$. Note that $X_1(t_i)$ need not equal $X_0(t_i)$; when $X_1(t_i) \neq X_0(t_i)$, $X_1(t)$ is said to be a "dispersed" trajectory relative to $X_0(t)$

$x_1(t)$ be the analytically derived trajectory with initial state $x_1(t_i) \equiv X_1(t_i)$

where $X_0(t)$, $x_0(t)$, $X_1(t)$, and $x_1(t)$ are n-vectors and t_i is the ith rectification time. Define the following new variables:

$$\rho_k(t) = X_k(t) - x_k(t), \qquad k=0,1 \tag{10}$$

$$\Delta\rho(t) = \rho_1(t) - \rho_0(t)$$
$$\Delta x(t) = x_1(t) - x_0(t) \tag{11}$$

Then from Fig. 2

$$X_k(t) = x_k(t) + \rho_k(t), \qquad k=0,1 \tag{12}$$

$$X_1(t) = X_0(t) - \rho_0(t) + \Delta x(t) + \rho_1(t) \tag{13}$$

Combination of Eqs. (10-13) yields

$$\boxed{X_1(t) = X_0(t) - x_0(t) + x_1(t) + \Delta\rho(t)} \tag{14}$$

Equation (14) is the trajectory combination equation for the Delta-Rho method. It relates the solution trajectory to the exact reference trajectory, two analytic trajectories, and the second-order term $\Delta\rho(t)$. Note that the first three terms of Eq. (14) can be determined as a function of time and combined prior to the evaluation of $\Delta\rho(t)$ which, as it turns out, is obtained by integration of the equations defining $\Delta\dot{\rho}(t)$. In the implementation of this method, the combination $X_0(t) - x_0(t)$ is formed prior to integration of $\Delta\dot{\rho}(t)$ and the result stored, while $x_1(t)$ is determined and combined with $\Delta\rho(t)$ during the integration process.

Assuming that $X_k(t)$ is defined such that it is in state-variable form, then

$$\dot{X}_k(t) = f[X_k(t)], \qquad k=0,1 \tag{15}$$

where $f[\cdot]$ is a vector function which must be (for this method) comprised of two vector functions $g[\cdot]$ and $\gamma[\cdot]$ such that

$$\dot{X}_k(t) = f[X_k(t)] = g[X_k(t)] + \gamma[X_k(t)], \qquad k=0,1 \tag{16}$$

$$\|\gamma[X_k(t)\| \ll \|g[X_k(t)]\|, \qquad k=0,1 \tag{17}$$

and with analytically determined n-vector states $x_k(t)$, $k=0,1$ which are known solutions to

$$\dot{x}_k(t) = g[x_k(t)], \qquad k=0,1 \tag{18}$$

$$x_k(t_i) \equiv X_k(t_i), \qquad k=0,1 \tag{19}$$

for a reference time t_i referred to as the rectification time. Differentiation and rearrangement of Eq. (14) together with substitution of Eqs. (16) and (18) yields

$$\Delta\dot{\rho}(t) = [\dot{X}_1(t) - \dot{X}_0(t)] - [\dot{x}_1(t) - \dot{x}_0(t)] \tag{20}$$

and

$$\boxed{\begin{aligned}\Delta\dot{\rho}(t) = {} & [g(X_1) - g(x_1)] - [g(X_0)] - g(x_0)] \\ & + [\gamma(X_1) - \gamma(X_0)]\end{aligned}} \tag{21}$$

where the explicit dependence on time has been omitted from the notation for convenience only. Equation (21) is the state equation for Delta-Rho.

The basic Delta-Rho method consists of the integration of Eq. (21) given the $X_k(t_i)$, $k=0,1$ n-vectors for some reference time t_i, and the algebraic addition of the resulting $\Delta\rho$ to the two analytically determined trajectories $x_k(t)$, $K=0,1$ and the stored exact reference trajectory $X_0(t)$ according to the trajectory combination equation (14).

This procedure can be rectified as required, that is if $\|\Delta\rho(t)\|$ becomes greater than some error criterion ϵ as the trajectories are propagated, the analytically determined trajectories $x_k(t)$, $k=0,1$ can be reinitialized to the exact trajectories $X_k(t)$, $k=0,1$ at rectification time t_{i+1} according to

$$x_k(t_{i+1}) = X_k(t_{i+1}), \qquad k=0,1 \tag{22}$$

for some $t_{i+1} > t_i$ such that $\|\Delta\rho(t_{i+1})\| < \epsilon$.

Rectification at time t_{i+1} reduces the magnitude of $\Delta\rho(t)$ for $t > t_{i+1}$ because $\rho_1(t_{i+1}) \equiv \rho_0(t_{i+1}) \equiv 0$ which results in $\Delta\rho(t_{i+1}) \equiv 0$ and consequently will generally enhance the accuracy of this method.

The $[g(X_0) - g(x_0)]$ and $\gamma(X_0)$ terms of Eq. (21) can be evaluated for required integration time points and stored prior to the integration of $\dot{\Delta}\rho(t)$ in a manner similar to that for the $[X_0(t) - x_0(t)]$ terms of Eq. (14).

The results to this point [Eqs. (1-22)] are quite general except for the restrictions defined by Eqs. (2), (3), (16), and (17) and can be applied to any continuous deterministic system for which these relations hold (i.e., trajectories which can be considered as perturbed trajectories with respect to an analytically determined trajectory). Application of these equations to astrodynamics will be presented in Sec. III and to a specific problem in Sec. IV.

C. Inclusion of General Perturbations in the Osculating Trajectories: The Generalized Delta-Rho Method

The approach which appears to offer the greatest performance improvement to both Encke's and the Delta-Rho method, in terms of increased accuracy and increased integration stepsize, is to refine the osculating trajectories by the use of a general perturbation theory (i.e., approximate analytic integration of the equations defining the perturbed trajectory). The value of this in astrodynamics has been shown by Kyner and Bennett,[4,5] Escobal,[6] and Born.[7]

Refinement of the osculating trajectories for the generalized Delta-Rho method can be readily accomplished if the following conditions are met. Assume that $X_k(t)$ is defined as in Sec. II.B, that is, that it is in state-variable form and that $\dot{X}_k(t)$ for $k=0,1$ can be expressed by Eq. (15), but in this case $f[\cdot]$ is a vector function which must be comprised of three vector functions $g[\cdot]$, $h[\cdot]$, and $\gamma[\cdot]$ such that

$$\dot{X}_k(t) = f[X_k(t)] = g[X_k(t) + h[X_k(t)] + \gamma[X_k(t)], \qquad k=0,1 \tag{23}$$

$$\| [h[X_k(t)] \| \ll \| g[X_i(t)] \|, \qquad k=0,1 \tag{24}$$

$$\| \gamma[X_k(t)] \| \ll \| g[X_i(t)] \|, \qquad k=0,1 \tag{25}$$

and with analytically determined n-vector states $y_k(t)$, $k=0,1$ which are known exact solutions to

$$\dot{y}_k(t) = g[y_k(t)], \qquad k=0,1 \tag{26}$$

$$y_k(t_i) \equiv X_k(t_i), \qquad k=0,1 \tag{27}$$

In addition, it is assumed that there are analytically determined n-vector states $x_k(t)$, $k=0,1$ which are known solutions to

$$\dot{x}_k(t) = g[x_k(t)] + h[x_k(t)] \tag{28}$$

$$x_k(t_i) \equiv X_k(t_i) \tag{29}$$

for a reference time t_i (the rectification time). In this case, however, the $x_k(t)$ trajectories can be either exact or approximate solutions to Eqs. (28) and (29). The $x_k(t)$ trajectories are the perturbed osculating trajectories which will, in most cases, be a useful refinement of the analytic exact solution trajectories $y_k(t)$ if the contribution of the $h[X_k(t)]$ perturbation term to $f[X_k(t)]$ is larger than that of $\gamma[X_k(t)]$.

Differentiation and rearrangement of Eq. (14) together with substitution of Eqs. (26) and (28) yields the state equation for the generalized Delta-Rho method,

$$\boxed{\begin{aligned}\dot{\Delta}\rho(t) = {} & [g(X_1) - g(X_0)] - [g(x_1) - g(x_0)] + [h(X_1) \\ & - h(x_1)] - [h(X_0) - h(x_0)] + [\gamma(X_1) - \gamma(X_0)]\end{aligned}} \tag{30}$$

where the explicit dependence on time has been omitted from the notation for convenience only. If

$$\| \gamma(X_k) \| \ll \| h(X_k) \|, \qquad k=0,1 \tag{31}$$

which is the case for the astrodynamics application investigated as part of this research, then

$$\| \gamma(X_1) - \gamma(X_0) \| \ll \| \dot{\Delta}\rho(t) \| \tag{32}$$

and the $[\gamma(X_1) - \gamma(X_0)]$ can be omitted from Eq. (30) yielding a simplified state equation,

$$\begin{aligned}\dot{\Delta}\rho(t) = {} & [g(X_1) - g(X_1)] - [g(x_1) - g(x_0)] \\ & + [h(X_1) - h(x_1)] - [h(X_0) - h(x_0)]\end{aligned} \tag{33}$$

The same trajectory combination equation (14) holds in this case. Equation (30) can replace Eq. (21) in the Delta-Rho procedure if Eqs. (24) and (25) hold and the corresponding perturbed osculating trajectory $x_k(t)$ is known. If Eq. (31) holds in addition to Eqs. (24) and (25), the simpler Eq. (33) can be used instead. As in the case of Eq. (21), Eqs. (30) and (33) are quite general and can be applied to any continuous deterministic system for which Eqs. (24), (25), and (31) (if appropriate) hold.

III. Formulations Applicable to Astrodynamics Trajectories

This section specializes the results of Sec. II to the usual orbit-integration problem of astrodynamics, treating Cowell's, Encke's, and the Delta-Rho methods in turn, and placing the similarities and differences into focus. A typical astrodynamics trajectory problem is the three-degree-of-freedom translational motion of a point mass in a central nonideal gravitational field with possible perturbative accelerations arising not only from the nonideal character of the central field but from other sources such as aerodynamic drag and solar radiation pressure. It is assumed that this motion is dominated by an inverse-square central term $g[X(t)]$ and a perturbing term $\gamma[X(t)]$ such as those shown in Eqs. (2) and (16), and possibly a "larger" perturbing term $h[X(t)]$ such as that in Eq. (23). The state vectors $X(t)$ and $x(t)$ are 6-vectors which are comprised of two 3-vectors, position $[R(t)$ and $r(t)]$ and velocity $[V(t)$ and $v(t)]$. Assume state variable representation

$$x(t) = \begin{bmatrix} r(t) \\ v(t) \end{bmatrix} \quad \text{and} \quad X(t) = \begin{bmatrix} R(t) \\ V(t) \end{bmatrix} \tag{34}$$

with state equations

$$\dot{x}(t) = g[x(t)] = \begin{bmatrix} \dot{r}(t) \\ \dot{v}(t) \end{bmatrix} \tag{35}$$

and

$$\dot{X}(t) = g[X(t)] + \gamma[X(t)] = \begin{bmatrix} \dot{R}(t) \\ \dot{V}(t) \end{bmatrix} \tag{36}$$

Now

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}[r(t)] = \dot{r}(t) \quad \text{and} \quad V(t) = \frac{\mathrm{d}}{\mathrm{d}t}[R(t)] = \dot{R}(t) \tag{37}$$

and the inverse-square central field part of the acceleration is given by

$$\dot{v} = -\frac{\mu}{r^3} r(t) \quad \text{and} \quad \dot{V} = -\frac{\mu}{R^3} R(t) \tag{38}$$

where $r^3 \equiv |r(t)|^3$, $R^3 \equiv |R(t)|^3$, and μ is the gravitational field constant.

Combination of Eqs. (34), (37), and (38) yield the astrodynamics matrix form of Eqs. (2) and (4). For the idea two-body osculating trajectories,

$$\dot{x}(t)=\begin{bmatrix}\dot{r}(t)\\ \dot{v}(t)\end{bmatrix}=g[x(t)]=\begin{bmatrix}0 & I_3\\ \frac{\mu}{r^3}I_3 & 0\end{bmatrix}\begin{bmatrix}r(t)\\ v(t)\end{bmatrix} \tag{39}$$

and for the perturbed "exact" trajectories

$$\dot{X}(t)=\begin{bmatrix}\dot{R}(t)\\ \dot{V}(t)\end{bmatrix}=g[X(t)]+\gamma[X(t)]$$

$$=\begin{bmatrix}0 & I_3\\ -\frac{\mu}{R^3}I_3 & 0\end{bmatrix}\begin{bmatrix}R(t)\\ V(t)\end{bmatrix}+\gamma[X(t)] \tag{40}$$

where I_3 is the 3×3 identity matrix. The vector function $h[X(t)]$ can be added if the conditions discussed in Sec. II.C apply.

A. Cowell's Method and Encke's Method

Cowell's method applied to astrodynamics consists of the numerical integration of Eq. (40). Encke's method applied to astrodynamics is readily obtained by substitution of the $g[\cdot]$ terms of Eq. (39) and (40) into Eq. (8),

$$\dot{\rho}(t)=\begin{bmatrix}0 & I_3\\ -\frac{\mu}{R^3}I_3 & 0\end{bmatrix}\begin{bmatrix}R(t)\\ V(t)\end{bmatrix}-\begin{bmatrix}0 & I_3\\ -\frac{\mu}{r^3}I_3 & 0\end{bmatrix}\begin{bmatrix}r(t)\\ v(t)\end{bmatrix}+\gamma[X(t)] \tag{41}$$

Let

$$\delta(t)\equiv R(t)-r(t) \qquad \rho(t)\equiv\begin{bmatrix}\delta(t)\\ \nu(t)\end{bmatrix}$$
$$\nu(t)\equiv V(t)-v(t) \tag{42}$$

Then Eq. (41) can be written in standard form[8,9]

$$\dot{\rho}(t)=\begin{bmatrix}\nu(t)\\ \frac{\mu}{r^3}\left[\left(1-\frac{r^3}{R^3}\right)R(t)-\delta(t)\right]\end{bmatrix}+\gamma[X(t)] \tag{43}$$

Since r^3/R^3 has a magnitude near one, numerical difficulties can arise when evaluating $1-r^3/R^3$. Procedures[8-12] are available to alleviate this problem. The procedure selected to be used with Delta-Rho is that described by Bate, Mueller, and White.[9] For the sake of compatibility, this procedure is also applied to Encke's method. Let

$$q(t)=-\frac{[\delta(t)]^T[r(t)+\frac{1}{2}\delta(t)]}{r^2(t)} \tag{44}$$

where $r^2(t)=|r(t)|^2$. The term $1-r^3/R^3$ can be expressed

$$\left(1-\frac{r^3}{R^3}\right)=1-[1-2q(t)]^{-3/2} \tag{45}$$

which when substituted into Eq. (43) yields the desired result,

$$\dot{\rho}(t)=\begin{bmatrix}\nu(t)\\ \frac{\mu}{r^3}\left[\{1-[1-2q(t)]^{-3/2}\}R(t)-\delta(t)\right]\end{bmatrix}+\gamma[X(t)] \tag{46}$$

Encke's method applied to astrodynamics consists of the integration of Eq. (46) to obtain $\rho(t)$ together with simultaneous evaluation of Eq. (9) to obtain $X(t)$. Evaluation of Eq. (9) requires the analytic solution of Eqs. (4) and (5) to obtain the osculating trajectory $x(t)$.

B. The Delta-Rho Method

In a similar manner, the Delta-Rho method applied to astrodynamics is obtained by substitution of the $g[\cdot]$ terms of Eqs. (39) and (40) into Eq. (21). Equation (21) becomes

$$\Delta\dot{\rho}(t)=\sum_{k=0}^{1}(-1)^{k+1}\left\{\begin{bmatrix}0 & I_3\\ -\frac{\mu}{R_k^3} & 0\end{bmatrix}\begin{bmatrix}R_k(t)\\ V_k(t)\end{bmatrix}-\begin{bmatrix}0 & I_3\\ -\frac{\mu}{r_k^3 I_3} & 0\end{bmatrix}\begin{bmatrix}r_k(t)\\ v_k(t)\end{bmatrix}\right\} \tag{47}$$

As in Encke's method, it is numerically advantageous to remove the difference of like magnitudes from Eq. (47). This is done in exactly the same manner as shown for the Encke's method except for the necessary addition of the subscript k. Let

$$q_k(t)=-\frac{[\delta_k(t)]^T[r_k(t)+\frac{1}{2}\delta_k(t)]}{r_k^2(t)}, \qquad k=0,1 \tag{48}$$

Then correspondingly

$$\Delta\dot{\rho}(t)=\sum_{k=0}^{1}(-1)^{k+1}\times\left\{\begin{bmatrix}\nu_k(t)\\ \frac{\mu}{r^3}\left[\{1-[1-2q_k(t)]^{-3/2}\}R_k(t)-\delta_k(t)\right]\end{bmatrix}+\gamma[X_k(t)]\right\} \tag{49}$$

The Delta-Rho method of astrodynamics consists of the integration of Eq. (49) to obtain $\Delta\rho(t)$ together with simultaneous evaluation of Eq. (14) to obtain the solution $X_1(t)$. Evaluation of Eq. (14) requires the analytic solution of Eqs. (18) and (19) to obtain the osculating trajectories $x_1(t)$ and $x_0(t)$. Rectification can be provided as appropriate. The generalized Delta-Rho method described in Sec. II.C is treated in the same manner with the appropriate combinations of the perturbation terms $h[\cdot]$ and $\gamma[\cdot]$ and the use of the appropriate osculating trajectories.

IV. Evaluation of the Delta-Rho Method

A. Program to Generate Standard Trajectories

One of the first steps in the implementation of a new technique is to establish a standard in order to compare and evaluate the performance of the new method. The candidates were selected from the special perturbation methods; general perturbation methods were excluded because their theory is based on applying approximation(s) to obtain analytic solutions, whereas the special methods numerically integrate equations containing perturbation terms as accurate as the models selected to represent them. In the method selected, the results can be made as accurate as required by choice of the computation precision and the integration stepsize. The comparisons of Conte,[13,14] Wolverton,[15] Roy,[10] Bate, Mueller, and White,[9] Baker,[11] and Brouwer and Clemence[16] were especially useful. Ultimately, a generalized variation-of-parameters (GVP) method, based on the work of Born,[17,18] Christiansen,[17-19] and Serversike,[18] was selected. It employs a universal-variable conic-trajectory propagation technique of Battin[20] and is effectively universal and free of singularities.

The standard trajectory program (also referred to herein as the GVP program) was coded in double-precision in order to eliminate the possibility of computer-related numerical errors. On the Burroughs B-6700 computer employed in the study, this gave 24 significant decimal digits. The sixth-order Runge-Kutta type of integration procedure of Luther[21] was adopted for numerical integration. While the code was general, based on the complex gravity recurrence relations of Cunningham,[23] and could have accommodated zonal, sectorial, and tesseral spherical harmonics out to the 22nd order, the tests of cases 1-3 were performed using spherical harmonics up to and including the fourth order. The GVP program was used to generate the reference trajectories required as standards with which to compare the Delta-Rho results. Complete details of the selection procedures and listings of key subroutines are presented on pp. 76-102 and Appendices A and B of Ref. 1.

B. Sample Cases

Development and numerical evaluation of the Delta-Rho method and its generalized variation was accomplished by selecting a specific astrodynamic problem and then experimenting with various numerical procedures to determine which yielded the best results. A reference coordinate system was adopted. This system is an inertial Earth-centered equatorial system with axes q^1, q^2, and q^3 such that q^1 and q^2 lie in the equatorial plane and q^3 is directed through the north pole. Physical constants were selected from a commonly used set.[22] For those trajectories requiring spherical harmonic coefficients not appearing in Ref. 22, values from Ref. 24 were used. The nominal trajectory (the exact reference trajectory) which was selected is a nearly minimum-energy ballistic trajectory with 7584.6 n. mi. Earth relative range and an initial inclination of 60 deg. This trajectory, together with particular data for its launch and impact points, is illustrated in Fig. 3. The initial state vector $X_0(t_0)$ for this trajectory is given in the first column of Table 1. This trajectory was selected because, while being of relatively short duration, it traverses the principal zonal and several of the tesseral harmonic regions defined by the oblate Earth spherical harmonic gravitational potential. The ballistic-type trajectory can potentially reduce computer run time requirements while providing an adequate test of the Delta-Rho method. Indeed, the apogee-passage characteristic of ballistic trajectories is likely to result in a stringent test of the algorithm's ability to keep in-track errors small.

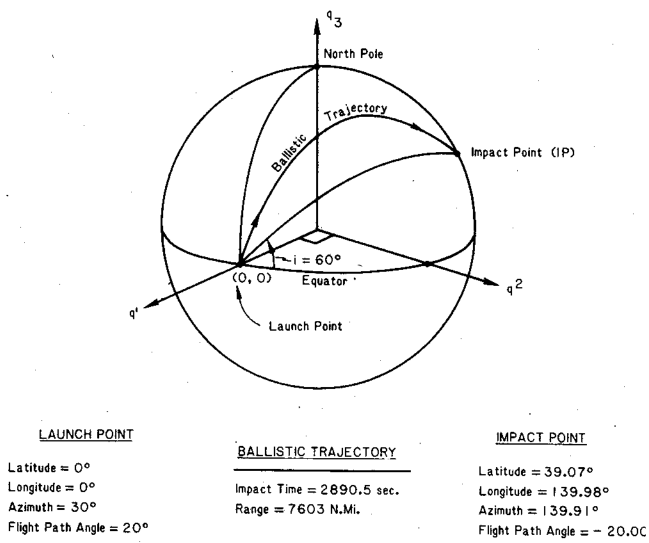

Fig. 3 Reference trajectory definition.

The initial conditions for three exact actual (solution) trajectories $X_I(t_0)$ were determined by incrementing the radius components of the initial conditions state vector $X_0(t_0)$ by 2000, 20,000, and 200,000 ft. The q^2 and q^3 velocity components were adjusted to restore the nominal impact point defined in Fig. 3, the q^1 velocity component remaining at its nominal value. Although the nominal trajectory has a duration of 2882.0 s from launch to impact, most comparisons were made at 2800.0 s. The initial conditions for the exact actual trajectories are given in the remaining columns of Table 1.

C. Simplification of Perturbation Terms in the Delta-Rho State Equation

Up to this point in the development, the very general spherical harmonic gravitational potential evaluation procedure of Cunningham[23] together with various orders of the gravitational model of Gaposchkin and Lambeck[24] were employed to define the perturbative acceleration in both the standard GVP and the Delta-Rho trajectory programs. Since the J_2 term of the gravitational potential is dominant and at least two orders of magnitude greater than any of the other spherical harmonic perturbation terms, it seemed reasonable to remove all the spherical harmonics except the J_2 term from the $\gamma(\cdot)$ perturbation terms appearing in Eq. (21). The $\gamma(\cdot)$ terms from which they have been removed appear as a difference in Eq. (21), so the removal of the higher order spherical harmonic terms should have a relatively minor

Table 1 Initial conditions for the reference and three dispersed trajectories

	Reference $X_0(t_0)$	Case 1 $X_I(t_0)$	Case 2 $X_I(t_0)$	Case 3 $X_I(t_0)$
	20,925,722.00	20,927,722.00	20,945,722.00	21,125,722.00
$R_k(t_0)$, ft	0.00	2,000.00	20,000.00	200,000.00
	0.00	2,000.00	20,000.00	200,000.00
	8772.758829	8772.758829	8772.58829	8772.758829
$V_k(t)$, ft/s	12051.47839	12050.78100	12044.14950	11977.53740
	20873.77287	20872.31260	20859.38870	20732.04110

Table 2 Numerical integration stepsize critical values and the required number of integration steps per trajectory

Integration scheme	Conservative stepsize critical value	Optimistic stepsize critical value	Conservative required no. of integration steps/trajectory	Optimistic required no. of integration steps/trajectory
RK(2-2)A	40	60	70.00→70	46.67→47
RK(2-2)B	60	80	46.67→47	35.00→35
RK(3-3)A	200	250	14.00→14	11.20→12
RK(3-3)B	70	90	40.00→40	31.11→32
RK(4-4)A	300	400	9.33→10	7.00→7
RK(4-4)B	300	350	9.33→10	8.00→8
RK(5-5)	600	700	4.67→5	4.00→4
RK(6-6)	500	750	5.60→6	3.73→4
RK(6-7)	500	750	5.60→6	3.73→4
RK(6-8)	800	1200	3.50→4	2.33→3
RK(7-7)	900	1200	3.11→4	2.33→3
RK(7-9)	700	1000	4.00→4	2.80→3
RK(8-10)	900	1300	3.11→4	2.15→3
RK(8-12)	900	1200	3.11→4	2.33→3

effect on the magnitude and integration of $\Delta\dot{\rho}$. However, these terms were retained in the exact reference trajectory $X_0(t)$ and in all trajectories used to evaluate errors in a $\Delta\rho$ trajectory.

This modification was made to the Delta-Rho computer program. Numerical verification indicated that the errors introduced by this simplification were acceptable and that the computer run time was reduced approximately 10%. For the ballistic cases defined in Table 2, the additional error contributed by this simplification was on the order of 0.5 ft on the trajectories of cases 1-3.

D. Investigation of Integration Rule: Results

Early indication of the feasibility of the Delta-Rho method led us to study the effect of choice of integration rule. There are two principal categories of numerical integration procedures, the one-step or single-step methods and the multistep methods. The one-step methods, such as those of the Runge-Kutta type, are typically easy to start, easy to terminate, and require little program logic in their corresponding codes. They do, however, require additional function evaluations per integration step which can be costly in terms of computer running time, depending on the complexity of the differential equations to be integrated. In addition, it is difficult to estimate the accumulated error over many steps. The multistep methods, on the other hand, typically require fewer function evaluations for an equivalent stepsize, but they can require iteration to achieve sufficient accuracy (e.g., predictor-corrector methods). In addition, these multistep methods must be "started," a procedure frequently accomplished by employing a one-step method to integrate the first step. Since the trajectories investigated are relatively short, few integration steps are required to span the trajectories, and since the simplicity of the code and the ease of starting and terminating a Runge-Kutta type of method is desirable for trajectory simulations of this type, Runge-Kutta methods were selected to be used in the various computer programs developed during this research.

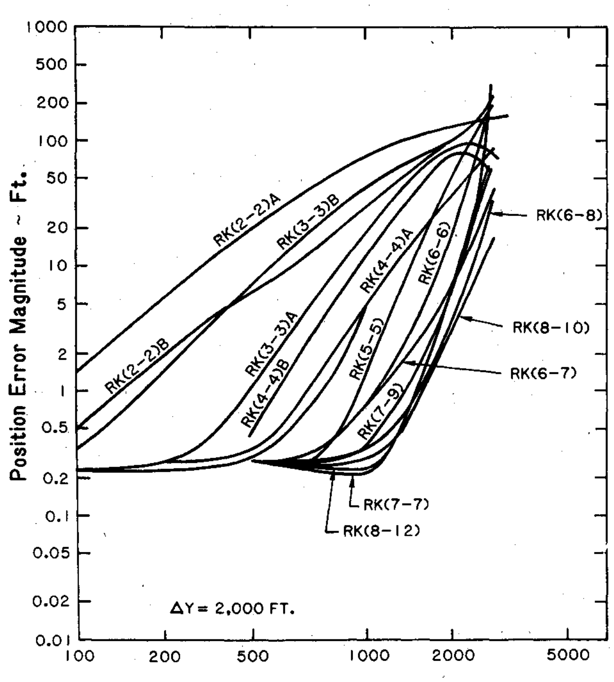

Fig. 4 Numerical integration accuracy, case 1.

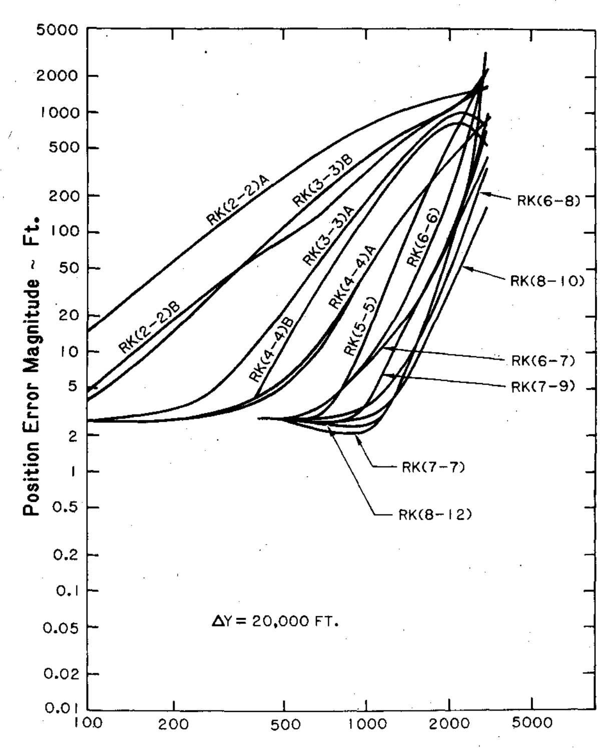

Fig. 5 Numerical integration accuracy, case 2.

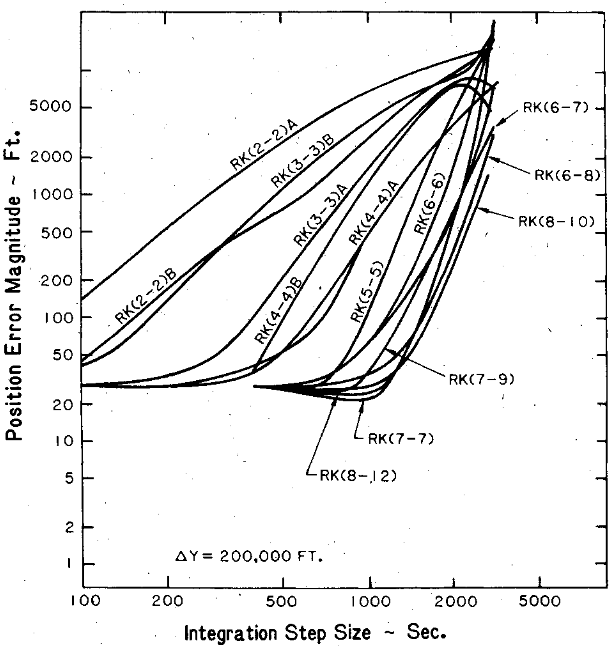

Fig. 6 Numerical integration accuracy, case 3.

For convenience, the various Runge-Kutta methods are identified by an expression of the form RK(i-j) where i defines the order of the method and j the required number of function evaluations per complete integration step. Methods ranging from RK(2-2) to RK(8-12) were selected. The basis for selection, as well as complete equations, are given on pp. 115-125 of Ref. 1.

The resulting comparisons for cases 1-3 are illustrated in Figs. 4-6, respectively. The position error magnitude is typically relatively independent of integration stepsize until it reaches a critical value, beyond which the error increases rapidly. In order to rank the individual integration procedures, it is necessary to identify this integration stepsize critical value. The criticial values for the various integration schemes are listed in Table 2. Two values for each scheme are shown. The first value, which is referred to as the conservative critical value, is defined prior to the elbow of the accuracy curves of Figs. 4-6. The second value, referred to as the optimistic critical value, is defined just after the elbow at a point where the accuracy would still be acceptable. The trajectory termination time was assumed to be 2800 s. In general this time occurs just prior to impact. The minimum number of integration steps required to span the trajectory is defined by the ratio of the 2800 s final time to the integration stepsize critical value. The actual minimum number is ob-

Table 3 Required number of function evaluations per trajectory

	Conservative		Optimistic	
Integration scheme	Required no. of function evaluations (fractional)	Required no. of function evaluations (integral)	Required no. of function evaluations (fractional)	Required no. of function evaluations (integral)
RK(2-2)A	141	141	95	95
RK(2-2)B	95	95	71	71
RK(3-3)A	43	43	35	37
RK(3-3)B	121	121	95	97
RK(4-4)A	39	41	29	29
RK(4-4)B	39	41	33	33
RK(5-5)	25	26	21	21
RK(6-6)	35	37	24	25
RK(6-7)	41	43	28	29
RK(6-8)	29	33	20	25
RK(7-7)	23	29	18	22
RK(7-9)	37	37	27	28
RK(8-10)	33	41	23	31
RK(8-12)	39	49	29	37

Table 4 Ranking of various Runge-Kutta numerical integration procedures

	Rankings					
	Conservative		Optimistic			
Integration scheme	Fractional	Integral	Fractional	Integral	Total points = Σ rankings	Overall ranking
RK(2-2)A	12	10	12	10	44	14
RK(2-2)B	10	8	11	9	38	12
RK(3-3)A	9	6	10	8	33	11
RK(3-3)B	11	9	12	11	43	13
RK(4-4)A	7	5	8	5	25	7
RK(4-4)B	7	5	9	7	28	9
RK(5-5)	2	1	3	1	7	2
RK(6-6)	5	4	5	3	17	4
RK(6-7)	8	6	7	5	26	8
RK(6-8)	3	3	2	3	11	3
RK(7-7)	1	2	1	2	6	1
RK(7-9)	6	4	6	4	20	6
RK(8-10)	4	5	4	6	19	5
RK(8-12)	7	7	8	8	30	10

tained by rounding this ratio up to the nearest integer. These values are listed in Table 2. The required number of function evaluations per trajectory is the product of the number of function evaluations per complete integration step and the appropriate required number of integration steps per trajectory plus one. Both the fractional (before round-up to the nearest integer) and the integral values, and both the conservative and the optimistic values of the required number of integration steps per trajectory from Table 2, are used to determine the values of the required number of function evaluations per trajectory as shown in the four data columns in Table 3. The integration methods can be ranked by the number of function evaluations per trajectory, the lower the number the higher the rank. This was done for the four data columns in Table 3 and the results are presented in the first four data columns of Table 4. The overall ranking, shown in data column six of Table 4, was obtained by ranking the sum of the individual rankings appearing in the first four columns (data column five).

This ranking procedure attempts to account for biases (occurring because a specific trajectory and termination time were assumed) by considering both the fractional and the integral values and both the conservative and the optimistic values. Although it is believed that this overall ranking is a good indication of the suitability of the various integration methods to the ballistic trajectory problem simulated by the Delta-Rho program, it is emphasized that this ranking procedure is just one of many reasonable ranking procedures that could have been used.

In general, the higher order (greater than four) integration methods performed better than the lower order methods for this problem. The Runge-Kutta-Shanks RK(7-7)[25,26] and RK(5-5) appear to be significantly better than the others and either should be used with the basic Delta-Rho computer program for trajectories of the type investigated.

The final form of the Delta-Rho program used Runge-Kutta-Shanks RK(7-7) for numerical integration, the gravitational potential evaluation procedure of Cunningham[23] truncated after J_2, and the universal variable method of Battin[20] for the propagation of the osculating trajectories.

E. Delta-Rho Method with Perturbed Osculating Trajectories

The use of perturbed osculating trajectories for the Delta-Rho method in accordance with the concepts of Sec. II.C was investigated. Verification that application of this concept would enhance accuracy and efficiency was accomplished numerically by using a GVP program to generate the "analytic" trajectories. The ballistic cases were examined in a manner similar to that used to investigate the effect of integration order and, for the same set of integration schemes, with the same range of stepsize values. Results showed that all of the methods could successfully span the trajectories with a single step with no significant loss in accuracy when the perturbed osculating trajectories were used. The resulting errors compare closely with the best results obtained with the basic Delta-Rho method even though they correspond to a single-step trajectory integration (i.e., a 2800 s stepsize). It is noted that these errors represent a lower attainable limit for the assumed perturbation models; the introduction of any analytically determined (and hence approximate) osculating trajectories can be expected to have greater errors.

One means to study the sensitivity to the introduction of a perturbation to the osculating trajectories is to define the osculating trajectories by using composite synthetic trajectories each one of which is comprised of a GVP J_2-only perturbed trajectory and an ideal two-body conic trajectory according to

$$\text{Osculating trajectory} \equiv \alpha\,[\text{GVP } J_2 \text{ trajectory}] + (1-\alpha)\,[\text{2-body conic trajectory}] \tag{50}$$

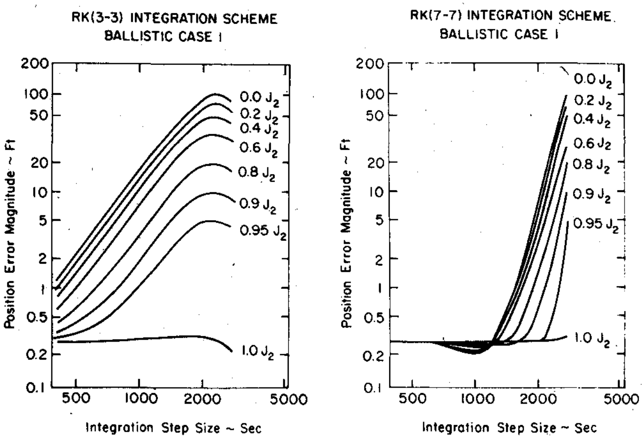

Fig. 7 Sensitivity to the degree of perturbation introduced into the osculating trajectories.

where $\alpha \in [0,1]$.

This method was used for case 1 by varying α from 0 to 1 and by using integration methods RK(3-3)A and RK(7-7). The results are presented in Fig. 7. They show the accuracy that is required in the perturbed osculating trajectory in terms of the fraction of $\alpha=1$ needed to achieve a desired integration stepsize and allowable position error magnitude. For example, using RK(7-7), a desired 2000 s integration stepsize will produce a position error of 2.5 ft in the selected case if the perturbed osculating trajectory spans 60% or more ($\alpha=0.6$) of the gap between the ideal two-body conic and an accurate J_2 trajectory.

F. General Comments

Cowell's method, the direct numerical integration of the equations of motion, is not a perturbation method at all. Encke's method is a true perturbation method; it integrates the difference ρ between the analytically determined osculating trajectory and the perturbed trajectory. Delta-Rho goes one step further in this hierarchy. It integrates the difference of the differences ($\Delta\rho = \rho_I - \rho_0$), as seen by comparing Eqs. (46) and (49). As such, it may be appropriate to refer to the Delta-Rho method as "an Encke on Encke."

This method is most useful when many accurate trajectories are needed in a common neighborhood. Such is the case in Monte Carlo and sensitivity studies (Appendix A). Another fruitful application may well be in the command post problem[27] where a computer onboard an aircraft must keep an accurate, real-time ephemeris for one or more satellites over several orbits, and where successive orbits can be considered to be dispersed orbits relative to the first. A third application may be in real-time guidance. In this application, a guidance law is developed to guide an actual vehicle along a trajectory which is a small perturbation from a nominal or reference trajectory (Appendix B).

Although the Delta-Rho method appears to have a high potential as an efficient perturbation technique, its accuracy diminishes as the relative magnitude of the perturbative acceleration approaches that of the primary term(s). Because of this characteristic, an auxiliary analytic method was developed specifically to treat nonperturbative applied accelerations such as thrust. This technique, called the analytic powered trajectory propagation (APTP) method,[28] was derived by curve-fitting functions which themselves are solutions to linear differential equations to those terms in the forcing function of the original state equations which were obstacles to their analytic solution. The resulting equations were linearized by expanding the dimension of the state to include the selected curve-fit functions as new state variables and then solved in a conventional manner. The resulting procedure was verified for a number of powered cases and found to provide sufficient accuracy to be used in conjunction with the Delta-Rho method when appropriate. To use APTP with Delta-Rho, one would include the nominal powered maneuver as part of the reference trajectory.

The Delta-Rho method is sensitive to the quality of the reference trajectory and to procedure, that is, program logic, sequence, and numerical methods. Good results can be obtained when the following suggestions are adhered to:

1) Optimally group and store the terms of the reference trajectory data.

2) Obtain and store the reference trajectory data at time points to be used in the numerical integration of the state equation.

3) Use the q method described above to preserve numerical accuracy in the small difference of large numbers.

To satisfy 1), store $[X_0(t) - x_0(t)]$ and $[g(X_0) - g(x_0)]$. The function $\gamma(X_0)$ may also be stored for convenience. Requirement 2 implies that the integration time points be known in advance; this limits selection of the integration methods to those having a constant stepsize. Failure to observe requirement 2 results in the need to interpolate between computed points; interpolation greatly degraded accuracy in other studies reported in Ref. 1. Although used, the provisions of requirement 3 were not needed on the Burroughs B6700 computer on which this study was performed, since it has 12 decimal digits of accuracy. Requirement 3 becomes necessary on computers such as the IBM 360, which has only six-decimal digit accuracy.

As pointed out earlier, excellent results are obtainable even when a greatly simplified perturbing force model is used in the Delta-Rho equations, provided that the full perturbing terms for requisite accuracy have been included in the reference trajectory $X_0(t)$.

V. Conclusions

A new trajectory perturbation technique, the Delta-Rho method, was developed and verified numerically for specific cases. This method is suitable for Monte Carlo studies, sensitivity analyses, real-time generation of ephemerides, and guidance applications, and appears to offer the prospects of significant computer time savings when so applied. While the basic version meets the research objectives, further improvements, including a single-step trajectory span capability, can be expected with the introduction of sufficiently accurate perturbed osculating trajectories. In addition, an analytic powered-trajectory propagation method was developed and verified. When used in conjunction with the Delta-Rho method, it will allow the inclusion of powered maneuvers for Monte Carlo and sensitivity analyses without the typically large increase in computer run time which occurs when conventional powered-trajectory simulation methods are employed.

In a given integration rule and stepsize, the endpoint position error of the Delta-Rho method varies linearly with the extent of the perturbation of the trajectory from the reference, down to essentially zero for a trajectory having zero initial perturbation. This behavior is fundamentally different than that which would be gotten using Encke's method, where very nearly the same terminal error would have been achieved on all three of the cases studied. With a 2000 ft initial perturbation, 1000 s stepsize, and RK(7-7) integration rule, a teminal error of 0.22 ft was obtained.

The Delta-Rho method can be applied to any continuous trajectory problem, in astrodynamics or in any other field, in which an approximate analytic solution is known and the "actual" solution considered a perturbation from the approximate.

Appendix A: Application of Delta-Rho to Monte Carlo Studies

Suppose, for example, that one desires to study the effect of boost phase variations on orbital insertion accuracy in an Earth launch to Earth orbit rendezvous mission. One would perform a detailed simulation of the boost phase of the mission, with the boost guidance law operative, varying such parameters as the thrust level of each vehicle, tailoff, drag, winds, etc., in Monte Carlo fashion, achieving a set of position-velocity 6-vectors at thrust termination. Where will each point of this set cause the vehicle to be, after it follows an orbital coast phase, at the nominal time of rendezvous? Rather than having to integrate each orbit accurately, one orbit can be integrated accurately; it can be used as the reference orbit. The location at the fixed rendezvous time of all other members of the set can then be obtained using Delta-Rho.

Appendix B: Application of Delta-Rho to Real-Time Guidance

Consider the onboard, real-time problem of a space ferry shuttling supplies and/or repair crews from one Earth orbit satellite to another. This mission in reality is a series of rendezvous trajectories. High accuracy is needed in the integration of orbits about the oblate Earth to achieve each rendezvous without wasting time or fuel. Let us assume that enough is known about each new rendezvous phase to develop an accurate nominal trajectory before that phase is begun. The nominal trajectory for the next phase could be developed, for example, in a large ground-based computer while the ferry is servicing its current space partner. The data from the reference trajectory would be radioed to the ferry and inserted into its onboard computer. The guidance law on the ferry, to determine, for example, when to cut off the long first burn of the new mission phase, could consist in part of the forward integration using Delta-Rho in faster than real time of the trajectory which will follow the imminent thrust termination, to find out where the vehicle will be at the nominal next rendezvous time or the nominal next burn time in a multiburn flight plan. It may also be desirable to include the APTP method of Sec IV.F for the remaining part of the present burn. The guidance law would then have to determine steering corrections to bring the future trajectory back to an acceptable set of end conditions. The principle being suggested follows that of the so-called "explicit" guidance laws illustrated in Ref. 29, with the difference being that whereas Ref. 29 computed coasting orbits as conics, the more sophisticated approach envisioned here would solve for coasting orbits as accurately perturbed conics.

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